

The four-dimensional Martínez Alonso–Shabat equation: reductions and nonlocal symmetries

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We consider the four-dimensional integrable Martínez Alonso–Shabat equation, and list three integrable three-dimensional reductions thereof. We also present a four-dimensional integrable modified Martínez Alonso–Shabat equation together with its Lax pair.

We also construct an infinite hierarchy of commuting nonlocal symmetries (and not just the shadows, as it is usually the case in the literature) for the Martínez Alonso–Shabat equation.

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1 Introduction

Consider the four-dimensional Martínez Alonso–Shabat equation

$$u_{ty} = u_z u_{xy} - u_y u_{xz} \quad (1)$$

introduced in [19]. It has [21] a covering defined by system

$$q_y = \lambda u_y q_x, \quad q_z = \lambda (u_z q_x - q_t) \quad (2)$$

with a non-removable parameter $\lambda \neq 0$, and a recursion operator, and is therefore integrable.

Below we present three reductions of (1) to integrable three-dimensional equations: the so-called rdDym equation [4, 25, 22, 24], the universal hierarchy equation [19], and an equation (4) related [2, 10] to the ABC equation, see [30, 7] for the latter.

Note that eliminating u from the Lax pair (2) for (1) yields an integrable four-dimensional PDE (10) to which we refer to as to the *modified* Martínez Alonso–Shabat equation and which can be seen as a four-dimensional generalization of the ABC equation (6). Integrability of (10) is established by presenting a Lax pair (11) for the latter. It would be interesting to study (10) in more detail, e.g. to find a recursion operator for this equation.

The main goal of the rest of the present paper is to find nonlocal symmetries for equation (1) in the covering (12) derived from the Lax pair (2).

Following [29, 17, 23] and references therein, recall that a *higher* (or *generalized* [23]) *symmetry* for a partial differential system \mathcal{E} can be identified with its characteristics, which is, roughly speaking, a vector function on an appropriate jet space $J^\infty(\mathcal{E})$ associated with \mathcal{E} that depends on finitely many arguments and satisfies the linearized version of \mathcal{E} .

Next, a (differential) *covering* for a partial differential system \mathcal{E} is (see e.g. [16, 17, 5, 15]) an overdetermined system $\tilde{\mathcal{E}}$ involving additional dependent variables (which are called *pseudopotentials*) such that \mathcal{E} implies the compatibility conditions of $\tilde{\mathcal{E}}$. A vector function on $J^\infty(\tilde{\mathcal{E}})$ that depends on finitely many arguments and satisfies the linearized version of \mathcal{E} is called [16, 18, 5, 15] a *shadow* (more precisely, a $\tilde{\mathcal{E}}$ -shadow) for \mathcal{E} .

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Thus, in contrast with the symmetry of \mathcal{E} , a shadow is allowed to depend on the pseudopotentials and their derivatives. On the other hand, symmetries of $\tilde{\mathcal{E}}$ are called [16, 18, 5, 17, 15] *nonlocal symmetries* of \mathcal{E} associated with the covering $\tilde{\mathcal{E}}$.

Infinite-dimensional Lie algebras of nonlocal symmetries are well known to play an important role in the theory of integrable systems and provide a useful tool for the study of the latter, see e.g. [4, 15] and references therein. In this connection note (see e.g. [5]) that not every $\tilde{\mathcal{E}}$ -shadow can be lifted to a full-fledged nonlocal symmetry for \mathcal{E} associated with $\tilde{\mathcal{E}}$. This fact has profound consequences. In particular, while for the full nonlocal symmetries we can readily define their Lie bracket, this is not quite the case for the shadows.

The results of Sections 4 and 5 of the present paper can now be summarized as follows.

First, using (2) as a starting point we construct a new covering (12) for (1). Expanding the pseudopotential in (12) into the formal Taylor series w.r.t. the spectral parameter λ gives a new covering (19) with an infinite number of new pseudopotentials w_i which are the coefficients at the powers of λ .

We then proceed to construct an infinite hierarchy of commuting nonlocal symmetries for equation (1) in this new covering using a technique from [26]. Let us stress that this construction makes heavy use of the fact that the covering (19) can be promoted to a covering (19)+(21b) over the system that consists of (1) and (21a). It should also be mentioned that finding full-fledged nonlocal symmetries for equations with more than two independent variables rather than mere shadows is quite uncommon, cf. e.g. [11, 26] and the discussion at the end of Section 5.

2 Reductions of the Martínez Alonso–Shabat equation

It is a remarkable fact that three known integrable three-dimensional PDEs can be obtained as reductions of (1).

2.1 The rdDym equation

The reduction $z = x$ yields the rdDym equation [4, 25, 22, 24] that arises as the $r \rightarrow \infty$ limit of the so-called r th dispersionless Harry Dym equation [4]:

$$u_{ty} = u_x u_{xy} - u_y u_{xx}. \quad (3)$$

The Lax pair (2) after the reduction boils down to the known Lax pair for (3),

$$q_t = (u_x - \lambda^{-1}) q_x, \quad q_y = \lambda u_y q_x.$$

2.2 The universal hierarchy equation

Putting $t = y$ in (1) and (2) yields the universal hierarchy equation [19]

$$u_{yy} = u_z u_{xy} - u_y u_{xz}$$

and its Lax pair

$$q_y = \lambda u_y q_x, \quad q_z = \lambda (u_z - \lambda u_y) q_x.$$

2.3 An equation related to the ABC equation

Another interesting reduction admitted by (1) arises when we put $z = t$. This produces the equation

$$u_{ty} = u_t u_{xy} - u_y u_{tx}, \quad (4)$$

which, along with the associated Lax representation,

$$q_t = \lambda u_t q_x / (\lambda + 1), \quad q_y = \lambda u_y q_x, \quad (5)$$

obtained by performing the reduction in question in (2), has already appeared in the literature, see [2, 10].

Upon eliminating u from (5) we arrive at the equation

$$q_y q_{tx} = (\lambda + 1) q_t q_{xy} - \lambda q_x q_{ty}, \quad (6)$$

which is, up to the removal of non-essential parameters, nothing but the so-called ABC equation

$$A q_x q_{ty} + B q_y q_{tx} + C q_t q_{xy} = 0, \quad A + B + C = 0, \quad (7)$$

which describes three-dimensional Veronese webs [30, 7]. This equation is also of importance in geometry: as shown in [9], to any smooth solution of (7) one can associate a three-dimensional Einstein–Weyl structure. The Bäcklund transformation (5) relating (4) and (6) appears in Remark 2 of [10]. In [6] it was shown that (7) with $A + B + C \neq 0$ is also integrable but has a nonisospectral Lax pair.

As a final remark, note that the equations (6) and

$$r_y r_{tx} = (\mu + 1) r_t r_{xy} - \mu r_x r_{ty}, \quad (8)$$

with the parameters $\mu \neq \lambda$ are related by the following Bäcklund transformation [30]:

$$r_t = \frac{\mu(\lambda + 1)}{\lambda(\mu + 1)} \frac{q_t}{q_x} r_x, \quad r_y = \frac{\mu}{\lambda} \frac{q_y}{q_x} r_x. \quad (9)$$

3 The modified Martínez Alonso–Shabat equation

Eliminating u from (2) yields an equation

$$q_y q_{xz} + \lambda q_x q_{ty} - (q_z + \lambda q_t) q_{xy} = 0, \quad (10)$$

which involves a parameter λ and can be considered as a four-dimensional generalization of the ABC equation (6) and reduces to the latter if $z = t$.

Moreover, there exists a 4D generalization of (9): for $\mu \neq \lambda$ the system

$$r_y = \frac{\mu}{\lambda} \frac{q_y}{q_x} r_x, \quad r_z = \frac{\mu}{\lambda} \frac{q_z + \lambda q_t}{q_x} r_x - \mu r_t \quad (11)$$

defines a Bäcklund transformation between equations (10) and

$$r_y r_{xz} + \mu r_x r_{ty} - (r_z + \mu r_t) r_{xy} = 0.$$

This *inter alia* means that (11) is compatible by virtue of (10), and thus (11) provides a Lax pair for (10) with the spectral parameter μ . Thus, (10) is a four-dimensional integrable equation, to which we refer as to the *modified Martínez Alonso–Shabat equation*.

4 New coverings for the Martínez Alonso–Shabat equation

Consider the Lax operators associated with the Lax pair (2),

$$L_1 = D_z + \lambda(D_t - u_z D_x), \quad L_2 = D_y - \lambda u_y D_x,$$

where D_t , D_x , D_y , D_z are total derivatives in the covering (2) over (1). We can now construct another covering over (1) as follows. Let $M = s D_x$. Then it is readily checked that the equations $[L_i, M] = 0$, $i = 1, 2$, boil down to the following equations for s :

$$s_y = \lambda(u_y s_x - u_{xy} s), \quad s_z = \lambda(u_z s_x - s_t - u_{xz} s). \quad (12)$$

These equations are compatible by virtue of (1), and thus define a covering over (1) with the pseudopotential s .

The covering (12) is of interest *inter alia* for the following reason: it is easily checked that $U = s$ satisfies the linearized version of (1)

$$U_{ty} = u_z U_{xy} - u_y U_{xz} + u_{xy} U_z - u_{xz} U_y, \quad (13)$$

i.e., s is a shadow for (1) in this covering.

To any shadow one can associate a covering in which this shadow is lifted to a nonlocal symmetry using the construction of [13, 14]. However, this requires an introduction of an additional infinite series of nonlocal variables, and these series are different for different shadows.

It is therefore a remarkable fact that for equation (1) we were able to construct a fairly simple covering (19) for which there exists an infinite commuting series of nonlocal symmetries of (1) expressible solely in terms of pseudopotentials w_i of this covering. The technique employed by us to this end below mimics the one from [26].

Namely, following [26], consider a copy of the covering (12) with λ replaced by another parameter μ and the pseudopotential denoted by w instead of s :

$$w_y = \mu (u_y w_x - u_{xy} w), \quad w_z = \mu (u_z w_x - w_t - u_{xz} w). \quad (14)$$

By the above, w is a shadow of nonlocal symmetry for (1). Informally this can be restated as follows: suppose that u , s , w also depend on an additional independent variable τ , then equation $u_\tau = w$ is compatible with (1), (14).

Moreover, it turns out that the system

$$u_\tau = w, \quad s_\tau = \frac{\lambda \mu}{\mu - \lambda} (w s_x - s w_x) \quad (15)$$

is compatible with (1), (12) by virtue of (1), (12), (14) and (15). It is important to stress that the extension to s of the first part of the flow, $u_\tau = w$, is not uniquely defined, see e.g. [5], i.e., the choice made for the right-hand side of the second equation of (15) is, generally speaking, not a canonical one.

Now slightly alter the notation to stress the dependence of all relevant objects on μ : write $\tau(\mu)$ instead of τ and $w(\mu)$ instead of w . In this notation $s = w(\lambda)$, so if we assume, following (15), that there holds

$$(w(\mu))_{\tau(\nu)} = \frac{\mu \nu}{\nu - \mu} (w(\nu) (w(\mu))_x - w(\mu) (w(\nu))_x), \quad (16)$$

and its counterpart with μ and ν interchanged, then the flows with the times $\tau(\mu)$ and $\tau(\nu)$ are readily checked to commute for all $\mu \neq \nu$:

$$\frac{\partial^2 u}{\partial \tau(\mu) \partial \tau(\nu)} = \frac{\partial^2 u}{\partial \tau(\nu) \partial \tau(\mu)}, \quad \frac{\partial^2 s}{\partial \tau(\mu) \partial \tau(\nu)} = \frac{\partial^2 s}{\partial \tau(\nu) \partial \tau(\mu)}. \quad (17)$$

As an aside note that if we put $\tilde{w}(\mu) = 1/(\mu w(\mu))$, then (16) can be written in the conservative form as

$$(\tilde{w}(\mu))_{\tau(\nu)} = \frac{\mu}{\nu - \mu} (\tilde{w}(\mu) \tilde{w}(\nu))_x. \quad (18)$$

Up to passing to the inverses of the parameters ($\mu' = 1/\mu$, $\nu' = 1/\nu$) and changing the sign of $\tau(\nu)$ and some bits of notation, this is nothing but equation (20) from [25], i.e., the equation for the generating function for the conserved densities of the so-called ε -system, see [25] for further details and cf. also [2, 28].

Expanding $w(\mu)$ into formal power series in μ , $w(\mu) = \sum_{i=0}^{\infty} w_i \mu^i$, now yields a new covering over (1) with the pseudopotentials w_i defined by the system

$$(w_i)_y = u_y (w_{i-1})_x - u_{xy} w_{i-1},$$

$$(w_i)_z = u_z (w_{i-1})_x - (w_{i-1})_t - u_{xz} w_{i-1} \quad (19)$$

for $i \in \mathbb{N}$ with an arbitrary smooth function $w_0 = w_0(t, x)$.

The pseudopotentials w_i , $i \in \mathbb{N}$, are readily checked to be shadows of nonlocal symmetries of equation (1) in the covering (19). It turns out that we reproduce the action of the inverse recursion operator \mathcal{R}^{-1} for (1) from [21]: we have $w_i = \mathcal{R}^{-1} w_{i-1}$, $i \in \mathbb{N}$.

5 Nonlocal symmetries of the Martínez Alonso–Shabat equation

We now intend to show that the shadows w_i can be lifted to full-fledged nonlocal symmetries for (1) in the sense of [16, 18, 5]. To this end we write, following the spirit of theory of generating functions for commuting flows, cf. e.g. equation (4) in [12], and references therein, a formal expansion

$$\frac{\partial}{\partial \tau(\mu)} = \sum_{i=0}^{\infty} \mu^i \frac{\partial}{\partial \tau_i},$$

and substitute this, along with the formal expansion for w in μ , into (15).

This results in the following equations:

$$u_{\tau_i} = w_i, \quad s_{\tau_i} = \sum_{k=0}^{i-1} \lambda^{k-i+1} ((w_k)_x s - w_k s_x), \quad i \in \mathbb{N}. \quad (20)$$

It is readily checked that for any given $i \in \mathbb{N}$ the system (20) is compatible with (1), (12) by virtue of (1), (12), (19) and (20).

As $s \equiv w(\lambda)$, we also have a formal expansion $s = \sum_{j=0}^{\infty} w_j \lambda^j$. Substituting this into the second equation of (20) yields the system

$$u_{\tau_i} = w_i, \quad (21a)$$

$$(w_j)_{\tau_i} = \sum_{k=0}^{i-1} (w_{i+j-k-1} (w_k)_x - w_k (w_{i+j-k-1})_x), \quad (21b)$$

which is compatible with (1) and (19); here $i, j \in \mathbb{N}$.

It is easily seen that the flows (21) commute, i.e., for all $i, j, k \in \mathbb{N}$ we have

$$\frac{\partial^2 u}{\partial \tau_i \partial \tau_j} = \frac{\partial^2 u}{\partial \tau_j \partial \tau_i}, \quad \frac{\partial^2 w_k}{\partial \tau_i \partial \tau_j} = \frac{\partial^2 w_k}{\partial \tau_j \partial \tau_i}. \quad (22)$$

In fact, this result can be extracted directly from (17).

Thus, we arrive at the following theorem, which can also be proved by direct computation instead of the above reasoning:

Theorem 1 *The infinite prolongations of the vector fields*

$$Q_i = w_i \frac{\partial}{\partial u} + \sum_{j=1}^{\infty} \sum_{k=0}^{i-1} (w_{i+j-k-1} (w_k)_x - w_k (w_{i+j-k-1})_x) \frac{\partial}{\partial w_j}, \quad (23)$$

where $i \in \mathbb{N}$, upon restriction to (1) and (19) form an infinite series of commuting nonlocal symmetries for equation (1) in the covering (19).

The commutativity of Q_i in this context means that the Lie brackets of the infinite prolongations of Q_i (or, equivalently, the Jacobi brackets of their characteristics, cf. [5]) vanish for all i . This essentially amounts to (22).

Note that existence of an infinite hierarchy of commuting flows is one of the most important hallmarks of integrability, see e.g. [8, 1, 17, 23, 3, 20] and references therein.

As we have $w_i = \mathcal{R}^{-1}w_{i-1}$, $i \in \mathbb{N}$, where \mathcal{R} is the recursion operator for (1) found in [21], cf. above, the commutativity (22) of the flows (20) suggests that \mathcal{R}^{-1} and \mathcal{R} should be hereditary (cf. e.g. [8, 3, 23] for more details on this property) in some appropriate sense, at least on the linear span of w_i . It is not quite clear, however, whether this claim can be made precise, let alone proved, because of the complicated structure of nonlocal terms in \mathcal{R}^{-1} and \mathcal{R} .

To conclude, let us stress again that finding an explicit form of the generators and the commutation relations for an infinite-dimensional algebra of nonlocal symmetries (rather than just the shadows, cf. e.g. [16, 18, 17, 5] for the details on differences among the two) for a non-overdetermined multidimensional PDE is quite rare. We were able to find just two similar results in the literature: the first is the commutative self-dual Yang–Mills hierarchy [1], and the second [11] is an infinite-dimensional noncommutative algebra of nonlocal symmetries for the (2+1)-dimensional integrable Boyer–Finley equation. It would therefore be very interesting to obtain the results similar to our Theorem 1 for other multidimensional integrable systems, for instance, those recently found in [27].

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